

Elliptic boundary value problems with Gaussian white noise loads

Sari Lasanen · Lassi Roininen · Janne M.J. Huttunen

Abstract Linear second order elliptic boundary value problems (BVP) on bounded Lipschitz domains are studied in the case of Gaussian white noise loads. Especially, Neumann and Robin BVPs are considered.

The main obstacle for applying the usual variational approach is that the Gaussian white noise has irregular realizations. In particular, the corresponding Neumann boundary values are not well-defined in the ordinary sense.

In this work, the BVP is reformulated by replacing the continuity of the boundary trace mapping with measurability. Instead of using variational methods alone, the reformulation of the BVP derives also from Cameron-Martin space techniques. The reformulation essentially returns the study of irregular white noise loads to study of L^2 -loads.

Admissibility of the reformulation is demonstrated by showing that usual finite element approximations of the BVP with discretized white noise loads converge to the solution of the reformulated problem. For Neumann and Robin BVPs, the finite dimensional approximations have been utilized before. However, here also the infinite-dimensional limit is considered.

Keywords Stochastic partial differential equations · Boundary value problems · Gaussian measures · Finite element methods · White noise

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1 Introduction

Let D be a bounded Lipschitz domain in \mathbb{R}^d , where the dimension $d \geq 2$, and denote with ∂_n the normal derivative at the boundary of D . We study a stochastic counterpart of a boundary

S. Lasanen
Mathematical Sciences, P.O.Box 3000, FI-90014 University of Oulu, Finland E-mail: sari.lasanen@oulu.fi

Lassi Roininen
Tallinn University of Technology, Department of Mathematics, Ehitajate tee 5, 19086 Tallinn, Estonia

Janne M.J. Huttunen, University of Eastern Finland, Department of Applied Physics, Yliopistonranta 1 F, FI-70211 Kuopio, Finland

value problem (BVP)

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } D \\ Bu = 0 & \text{on } \partial D, \end{cases} \quad (1)$$

where $f \in H^{-1}(D)$ is replaced with the Gaussian zero mean white noise \dot{W} on D and the boundary operator B is either of the Dirichlet type ($Bu = u|_{\partial D}$), Neumann type ($Bu = \partial_n u|_{\partial D}$), or the Robin type ($Bu = \partial_n u|_{\partial D} + \beta u|_{\partial D}$ for $\beta \in \mathbb{R}$). The constant λ in (1) is positive for simplicity.

The study of stochastic elliptic boundary value problems (1) initiated from works of Walsh [26, 27] who considered solvability of the Poisson equation with zero Dirichlet boundary condition and the white noise source. Walsh studied the very weak formulation of (1) in the sense of generalized functions, that is, distributions.

In the case of the homogeneous Dirichlet boundary condition $Bu = u|_{\partial D} = 0$ and the white noise load $f = \dot{W}$, the existence and uniqueness of pathwise continuous solution of (1) is well-known for $d = 1, 2, 3$, even for nonlinear equations by results of Buckdahn and Pardoux [9]. The corresponding Neumann and Robin problems are less extensively studied, although there are numerous studies on elliptic BVPs with more regular deterministic loads. This leaves a gap in the literature which appears for example in connection with Bayesian statistical inverse problems, where solutions of stochastic BVPs serve as priors [21]. The aim of this work is to provide a rigorous description of the stochastic BVP with the white noise load that utilizes both the stochastic nature of the problem and the existing literature on more regular problems.

The main difference between stochastic Dirichlet and Neumann problems is the specification of the solution space. In [9], one seeks a stochastic field X with continuous realizations that satisfies $-\Delta X + \lambda X = \dot{W}$ in the sense of distributions, and $X|_{\partial D} = 0$. In corresponding Neumann and Robin problems the normal derivative at the boundary is not well-defined when only continuity of the realizations has been verified, which is the main obstacle for formulating the problem. Indeed, the weak definition of the (co)normal derivative $\partial_n u$ at the boundary ∂D of the variational solution u of (1) requires that functionals

$$\int_{\partial D} \phi \partial_n u d\sigma := \int_D \nabla u \cdot \nabla \phi + \lambda u \phi - f \phi dx \quad (2)$$

are well-defined for all ϕ in a suitable function space H , which e.g. is satisfied when $u \in H^1(D) =: H$ and $f \in L^2(D)$ (see for example [19]).

There are several studies on how to extend an elliptic BVP to irregular loads or irregular boundary data. In [2] and references therein, BVPs are taken to be deterministic with no loads but highly irregular boundary values. Obviously, the above problem can be cast in such a form. For smooth boundaries, the several proposed extensions in [2] work nicely but for polygonal domains turn out to be problematic. A similar theme can be found in [4]. Rozanov [22] treats random fields as Hilbert space processes, and applies theory of distributions in defining the boundary traces for C^2 -smooth boundaries. Smoothness of boundaries benefits the definition of distributions on the boundary. An attempt to solve the Neumann boundary value problem with the help of Lax-Milgram theorem is made in [15]. However, the paper does not take into account that some of the stochastically integrated functions are anticipating which suggests that correct formulation would involve multidimensional Skorohod integrals. Also the interpretation of the normal derivative is left vague. A correct formulation with more regular loads can be found e.g. in [3, 25], but it is clear that the white noise loads do not fulfill the required conditions. The work in [15] can be appreciated from the

point of view of a more pragmatic question, which asks whether the white noise could be approximated by more regular stochastic fields (in the sense of an existing limit).

The approximations of white noise in Dirichlet problems are often carried out together with finite element methods [1, 5, 10, 11, 15, 25, 28]. Also, the convergence of approximative solutions has been verified [1, 5, 10, 11, 25, 28].

In [9], the homogeneous Dirichlet solution is acquired by replacing the BVP with a Hammerstein integral equation. A similar integral equation could be written in the Neumann or Robin case (see [12]) by updating the Dirichlet Green's function $G(x, y)$ with a correct boundary value. In the linear case, the conjectured integral equation would be $X(x) + \lambda \int_D G(x, y) X(y) dy = \int_D G(x, y) dW_y$ for a stochastic field X with a.s. continuous realizations, where dW_y represents multidimensional Itô integral. This is referred to as the mild form of the problem. However, it is not clear whether the realizations of X would fulfill the boundary condition $\partial_n X|_{\partial D} = 0$ in any other than mild sense. For smooth domains, a partial answer can be found in [2] for the description of the BVP, where such a formulation is compared to another generalization of irregular boundary values.

We proceed in different direction than in [2, 4]. Instead of trying to stretch the definition of the differentiability, we stretch the definition of the boundary trace with measure theoretic methods. Indeed, replacing f in (2) with L^2 -approximations of the white noise hints that a rigorous definition of the normal derivative of X might not call for continuity of the linear forms (2) on $H^1(D)$ but only measurability. Similar phenomenon appears in the variational formulations of BVPs with different boundary conditions. For $f \in L^2(D)$, the variational form of the homogeneous Dirichlet BVP is to find $u \in H_0^1(D)$ that satisfies

$$\int_D f(x) \psi(x) dx = \int_D \nabla u(x) \cdot \nabla \psi(x) + \lambda u(x) \psi(x) dx \quad (3)$$

for all $\psi \in H_0^1(D)$. Replacing f with regular approximations of white noise hints again to measurability of the linear forms. Indeed, in the case of homogeneous Dirichlet problem, such approximative variational solutions are known to converge in $L^2(\Omega, \Sigma, P; L^2(D))$ -norm to the correct solution [10]. The corresponding limit of the variational equations, when refining white noise approximations is then

$$\int_D \psi(x) dW_x = \lim_{n \rightarrow \infty} \int_D \nabla X_n \cdot \nabla \psi + \lambda X_n \psi dx \quad (4)$$

for every $\psi \in H_0^1(D)$, where dW_x represents multidimensional Itô integral and X_n are variational solutions of (3) with the approximated white noise.

The present paper contributes in this area by giving explicit formula for the normal derivative of the solution of (1) as a measurable mapping (see Definition 5). Instead of tackling directly the variational formulations of general BVP or trying to interpret the normal derivative in distributional sense, we reformulate the irregular elliptic BVP so that existing results for more regular elliptic BVP can be easily utilized. The approach also avoids the need to provide new estimates for the corresponding Green's functions, as is often the case in mild formulations. For example, continuity of the solution of two and three dimensional Neumann and Robin problems with the white noise load follows from the regularity of the deterministic problem via well-known Gaussian arguments. Moreover, unique solvability of high-dimensional problems is also guaranteed.

The reformulation of BVP involves Cameron-Martin space techniques. The main tool is the method of extending continuous linear mappings L on the Cameron-Martin space of a Gaussian field $X \sim \mathcal{N}(0, C_X)$ into measurable linear mappings \hat{L} on the sample space of X

(see e.g. [6]). Measurable linear extensions are applied in defining the boundary operators for the Gaussian fields. T

In order to demonstrate admissibility of the reformulation, we show that finite element approximations X_n of the solution X converge to the solution of the problem. The proof reduces essentially to a one-liner (23), even for high-dimensional problems.

The main approach to finite element methods (FEM) with irregular stochastic loads was introduced in [1], where the stochastic load f is first approximated by a spatially piecewise constant function, and then the ordinary FEM is applied (see also [10, 15]). However, even in 1D case the solutions of (1) with the white noise load are not regular enough for standard pathwise error methods [1]. The convergence of FEM approximations is therefore recast as a question of convergence of random variables, where several other modes of convergence are available besides to pathwise convergence. From previous studies [10] it is known that random fields X_n converges to X in norm

$$\|X\| := \left(E \|X\|_{L^2(D)}^2 \right)^{\frac{1}{2}} \quad (5)$$

for 2D Dirichlet problem. In [11], 3D case on a convex smooth domain is considered. Also, estimates for the speed of convergence are known [10]. Other similar works are [1, 14, 15, 28]. More regular loads are considered in [3, 25].

We improve the previous results by replacing $L^2(D)$ with $C(D)$ and giving generalization to cases of Neumann and Robin boundary data. The cases of Neumann and Robin boundary conditions are new.

The contents of this paper is the following. In Section 2 we recall known results about Gaussian random variables and their linear transformations. In Section 3 we define the measurable boundary trace and measurable normal derivatives (see Definition 5). In Section 4 we formulate the BVP and study its unique solvability. In Section 5 the regularity of the solutions is considered. In Section 6 the finite element approximations are studied.

2 Measure theoretic preliminaries

Let (Ω, Σ, P) denote a complete probability space. We make a standing assumption that all random variables are defined on (Ω, Σ, P) . Moreover, all appropriate function spaces appearing below are endowed with their Borel σ -algebras. We will denote with $H^s(D)$, $s \in \mathbb{R}$, the usual Sobolev spaces on D and with $H_0^s(D)$ the usual closure of compactly supported smooth functions on D (see e.g. [19]).

In this work, we extensively use the theory of Gaussian function-valued random variables and their linear functionals. As an introduction to present ideology, we recall the basic definitions in the case of white noise.

Let $\mathcal{B}(D)$ denote the Borel sets of a bounded Lipschitz domain $D \subset \mathbb{R}^d$. Recall, that \dot{W} is the white noise on D if $\{\dot{W}(A) : A \in \mathcal{B}(D)\}$ are Gaussian random variables with zero mean and covariance $\mathbb{E}\dot{W}(A)\dot{W}(B) = |A \cap B|$, where $|A|$ denotes the Lebesgue's measure of A , and $\dot{W}(A \cup B) = \dot{W}(A) + \dot{W}(B)$ a.s. for disjoint A and $B \in \mathcal{B}(D)$. A common way to construct functionals $\dot{W}(A)$ is through stochastic integrals

$$\dot{W}(A) = \int 1_A(x) dW_x,$$

with respect to d -dimensional Wiener field W_x , which is a Gaussian field with zero mean and covariance $\mathbb{E}W_x W_y = \min(x_1, y_1) \cdots \min(x_d, y_d)$ for all $x = (x_1, \dots, x_d), y = (y_1, \dots, y_d) \in \mathbb{R}^d$.

The Itô isometry allows us to replace characteristic functions 1_A of Borel sets $A \in \mathcal{B}(D)$ by functions $\phi \in L^2(D)$, and hence define white noise functionals

$$\dot{W}(\phi) := \int_D \phi(x) dW_x \quad (6)$$

as Gaussian random variables with zero mean and variance $\mathbb{E}\dot{W}(\phi)^2 = \|\phi\|_{L^2(D)}^2$.

Instead of considering solutions for the elliptic boundary value problem as stochastic fields, we take the more general approach by considering solutions (and the white noise) as Banach space valued random variables. Let us recall some definitions (e.g. [6, 7]).

Let \mathbb{B} be a separable Banach space. A mapping $X : \Omega \rightarrow \mathbb{B}$ is a \mathbb{B} -valued random variable if $X^{-1}(A) \in \Sigma$ for all Borel sets $A \subset \mathbb{B}$. Denote $\mu_X = P \circ X^{-1}$ the image measure of X on \mathbb{B} . Let \mathbb{B}^* denote the topological dual of \mathbb{B} and $\langle \cdot, \cdot \rangle_{\mathbb{B}, \mathbb{B}^*}$ denote the duality. A \mathbb{B} -valued random variable X is called Gaussian if $\langle X, b^* \rangle_{\mathbb{B}, \mathbb{B}^*}$ is Gaussian for all $b^* \in \mathbb{B}^*$.

For notational simplicity, we focus on reflexive \mathbb{B} . In the case of reflexive \mathbb{B} , we denote with $m \in \mathbb{B}$ the mean of X i.e.

$$\langle m, b^* \rangle_{\mathbb{B}, \mathbb{B}^*} = \mathbb{E} \langle X, b^* \rangle_{\mathbb{B}, \mathbb{B}^*}$$

for all $b^* \in \mathbb{B}^*$ and with $C_X : \mathbb{B}^* \rightarrow \mathbb{B}$ the covariance operator of X i.e.

$$\langle C_X b^*, b^* \rangle_{\mathbb{B}, \mathbb{B}^*} = \mathbb{E} \langle X - m, b^* \rangle_{\mathbb{B}, \mathbb{B}^*} \langle X - m, b^* \rangle_{\mathbb{B}, \mathbb{B}^*}$$

for all $b^* \in \mathbb{B}^*$.

Next, we recall that the white noise \dot{W} is $H^{-d/2-\varepsilon}(D)$ -valued Gaussian random variable for any $\varepsilon > 0$. Indeed, almost sure realization properties of the d -dimensional white noise can be derived by using random functionals $\dot{W}(\phi)$. The definition of the stochastic integral helps in identifying realizations of the white noise as weak derivatives of realizations of the Wiener field. Then the random functional $\dot{W}(\phi)$ can be identified with the linear functional $\langle \dot{W}, \phi \rangle$ between a distribution \dot{W} and a test function $\phi \in C_0^\infty(D)$. It is an easy task to apply duality to study Sobolev norms $\|\dot{W}\|_{H^{-s}(D)} = (\sum_{k=1}^\infty |\langle \dot{W}, \phi_k \rangle_{H^{-s}(D), H^s(D)}|^2)^{\frac{1}{2}}$ where $\phi_k \in C_0^\infty(D)$ form an orthonormal basis in $H_0^s(D)$ and $s \geq 0$. Recall, that orthonormal basis can be chosen from a dense set, and the dual of $H^{-s}(D)$ can be identified with $H_0^s(D)$ (see Theorem 3.30 in [19]). In particular, \dot{W} belongs a.s. to the Sobolev space $H^{-d/2-\varepsilon}(D)$ for any $\varepsilon > 0$, since the series of variances $\sum_{k=1}^\infty \|\phi_k\|_{L_2(D)}^2$ converges (see [17], Theorem 2 in Chapter. 3) by Maurin's theorem (e.g. [13]). Similarly, \dot{W} belongs to $H^{-d/2}(D)$ with probability zero.

The measurability of white noise can be checked by the well-known Pettis' measurability theorem, which says that a \mathbb{B} -valued mapping is a \mathbb{B} -valued random variable, if it is weakly measurable i.e. mappings $\langle X, b^* \rangle$ are random variables for all $b^* \in \mathbb{B}^*$. Hence, \dot{W} is $H^{-d/2-\varepsilon}(D)$ -valued Gaussian random variable for any $\varepsilon > 0$.

White noise \dot{W} has mean zero and identity as the covariance operator.

Definition 1 Let \mathbb{B} be a separable reflexive Banach space and let X be a Gaussian \mathbb{B} -valued zero mean random variable whose covariance operator C_X is nontrivial. Set

$$\|b^*\|_{\mu_X} := \sqrt{\langle C_X b^*, b^* \rangle_{\mathbb{B}, \mathbb{B}^*}}$$

for all $b^* \in \mathbb{B}^*$ and denote with $\mathbb{B}_{\mu_X}^*$ the closure of \mathbb{B}^* in norm the $\|\cdot\|_{\mu_X}$.

It is well-known that the elements of $\mathbb{B}_{\mu_X}^*$ can be identified with μ_X -measurable linear functionals on \mathbb{B} . More precisely, a functional on \mathbb{B} is a μ -measurable linear functional if it is μ -measurable and it has a version that is linear on a linear subspace of full μ -measure. The measurability of $\hat{h} \in \mathbb{B}_{\mu_X}^*$ can be seen as follows. For every $\hat{h} \in \mathbb{B}_{\mu_X}^*$ there exists a sequence $\{b_k^*\} \subset \mathbb{B}^*$ so that $\lim_{k \rightarrow \infty} b_k^* = \hat{h}$ in \mathbb{B}_{μ_X} . But then the linear functionals

$$b \mapsto \langle b, b_k^* \rangle_{\mathbb{B}, \mathbb{B}^*}, k \in \mathbb{N},$$

form a Cauchy sequence in $L^2(\mu_X)$. By taking a suitable subsequence, we obtain μ_X -a.s. limit

$$b \mapsto \lim_{j \rightarrow \infty} \langle b, b_{k_j}^* \rangle_{\mathbb{B}, \mathbb{B}^*} =: \hat{h}(b) \quad (7)$$

and

$$\|\hat{h} - b_{k_j}\|_{\mu_X}^2 = \mathbb{E} \left(\hat{h}(X) - \langle X, b_{k_j}^* \rangle_{\mathbb{B}, \mathbb{B}^*} \right)^2.$$

Each $\hat{h} \in \mathbb{B}_{\mu_X}^*$ defines a measurable functional (7) which are linear on a full measure linear subspace (for details, see Theorem 2.10.9 and Theorem 3.2.3 in [6]). We summarize the above facts in the next lemma.

Lemma 1 *The elements \hat{h} of $\mathbb{B}_{\mu_X}^*$ can be identified with μ_X -measurable linear functionals $b \mapsto \hat{h}(b)$ that are Gaussian zero mean random variables on the Lebesgue's completion of the probability space $(\mathbb{B}, \mathcal{B}(\mathbb{B}), \mu_X)$. Moreover, the covariance*

$$(\hat{h}, \hat{g})_{\mu_X} := \left(\int \hat{h}(b) \hat{g}(b) \mu_X(db) \right),$$

where $\hat{h}, \hat{g} \in \mathbb{B}_{\mu_X}^*$, defines an inner product on $\mathbb{B}_{\mu_X}^*$ and $(\hat{h}, \hat{h})_{\mu_X} = \|\hat{h}\|_{\mu_X}^2$.

The difference between just measurable and a measurable linear functional is that the versions of measurable linear functionals are not allowed to be modified on arbitrary null sets but only those null sets that will not destroy the linearity.

Especially, the mapping

$$B_{\mu_X}^* \times B^* \ni (\hat{h}, b_2^*) \mapsto \mathbb{E} \hat{h}(X) \langle X, b_2^* \rangle_{\mathbb{B}, \mathbb{B}^*}$$

is bilinear, and by Fernique's theorem (see e.g. [6]) bounded in the sense that

$$|\mathbb{E} \hat{h}(X) \langle X, b^* \rangle_{\mathbb{B}, \mathbb{B}^*}| \leq \left(\mathbb{E} \hat{h}(X)^2 \right)^{\frac{1}{2}} \left(\mathbb{E} \|X\|_{\mathbb{B}}^2 \|b^*\|_{\mathbb{B}^*}^2 \right)^{\frac{1}{2}} \leq C \|\hat{h}\|_{\mu_X} \|b^*\|_{\mathbb{B}^*}. \quad (8)$$

Remark 1 By (8) and reflexivity of \mathbb{B} , we may extend the covariance operator $C_X : \mathbb{B}^* \rightarrow \mathbb{B}$ as a continuous mapping from $\mathbb{B}_{\mu_X}^*$ to \mathbb{B} , and we continue to denote the extension with C_X i.e

$$\langle C_X \hat{h}, b^* \rangle_{\mathbb{B}, \mathbb{B}^*} = \mathbb{E} \hat{h}(X) \langle X, b^* \rangle_{\mathbb{B}, \mathbb{B}^*} \quad (9)$$

for all $\hat{h} \in B_{\mu_X}^*$ and $b^* \in B^*$.

Definition 2 Let X, C_X, \mathbb{B} , and $\mathbb{B}_{\mu_X}^*$ be as in Definition 1 and extend C_X as in Remark 1. The *Cameron-Martin space of X* is the set

$$H_{\mu_X} = C_X(\mathbb{B}_{\mu_X}^*)$$

equipped with the inner product

$$(h, g)_{H_{\mu_X}} = \int_B \widehat{h}(b) \widehat{g}(b) \mu_X(db), \quad (10)$$

where for all $h \in H_{\mu_X}$ the notation \widehat{h} means such a vector in $\mathbb{B}_{\mu_X}^*$ that $C_X \widehat{h} = h$. The corresponding inner product norm is denoted with $\|h\|_{H_{\mu_X}}$.

Remark 2 The Cameron-Martin space H_{μ_X} and the space of measurable linear functionals $\mathbb{B}_{\mu_X}^*$ are separable Hilbert spaces for all Gaussian random variables X that have values in separable Banach spaces (see Theorem 3.2.7 in [6]).

Remark 3 The covariance operator $C_X : \mathbb{B}_{\mu_X}^* \rightarrow H_{\mu_X}$ is an isometric isomorphism. From the inner product (10), we derive the bilinear form

$$\langle h, \widehat{g} \rangle_{H_{\mu_X}, \mathbb{B}_{\mu_X}^*} = \langle C_X \widehat{h}, \widehat{g} \rangle_{H_{\mu_X}, \mathbb{B}_{\mu_X}^*} := \int_B \widehat{h}(b) \widehat{g}(b) \mu_X(db),$$

and thus identify $\mathbb{B}_{\mu_X}^*$ as the dual space of the Cameron-Martin space. By (7) and (9),

$$\langle h, \widehat{g} \rangle_{H_{\mu_X}, \mathbb{B}_{\mu_X}^*} = \widehat{g}(C_X \widehat{h}) = \widehat{g}(h), \quad (11)$$

for proper linear versions of $b \mapsto \widehat{g}(b)$ since the Cameron-Martin space is contained in every linear subspace of full measure (see [6], Theorem 2.4.7). By density, we may always choose an orthonormal basis $\{\widehat{e}_k\}$ of $\mathbb{B}_{\mu_X}^*$ that consists of functions in \mathbb{B}^* and the corresponding image $C_X \widehat{e}_k \subset \mathbb{B}$ is an orthonormal basis of the Cameron-Martin space

We recall that the Cameron-Martin space of W is $L^2(D)$. In general, the Cameron-Martin space of a \mathbb{B} -valued random variable is separable and the Cameron-Martin space does not depend on the sample space of the \mathbb{B} -valued random variable X (see Theorem 3.2.7 and Lemma 3.2.2 in [6]). Moreover, since the zero-mean Gaussian X has values in \mathbb{B} , then the inclusion mapping of the Cameron-Martin space into \mathbb{B} is Hilbert-Schmidt (see [6], Corollary 3.5.11).

Let us recall the definition of measurable linear operator in our setting (see Definition 3.7.1 in [6] for a more general formulation).

Definition 3 Let $\mathbb{B}_1, \mathbb{B}_2$ be separable Banach spaces equipped with their Borel σ -algebras and let μ be a Borel probability measure on \mathbb{B}_1 . A mapping $T : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ is a μ -measurable linear operator if there exists a linear mapping $S : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ such that S is μ -measurable and $S = T$ μ -a.e. The linear mapping S is called a *proper linear version of T* .

In the case of Gaussian measures, there is a close relationship between measurable linear operators and Gaussian random series (see [7], Theorem 1.4.5 and Corollary 1.4.6-7). In the next theorem, we explicitly state the form of the measurable linear operators (the result is a minor modification of [7], Corollary 1.4.6).

Theorem 1 *Let X be a zero mean Gaussian random variable on a separable reflexive Banach space \mathbb{B}_1 , let H_{μ_X} denote the Cameron-Martin space of X and let $\{\widehat{e}_k\}_{k=1}^\infty \subset \mathbb{B}^*$ denotes an orthonormal basis of $\mathbb{B}_{\mu_X}^*$. If T is a continuous linear mapping from H_{μ_X} into a separable Hilbert space \mathbb{H} , then*

$$\widehat{T}(b) = \sum_{k=1}^{\infty} \widehat{e}_k(b) T C_X \widehat{e}_k \quad (12)$$

defines a μ_X -measurable linear operator $\widehat{T} : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ for any separable Hilbert space \mathbb{B}_2 such that the inclusion mapping $\mathbb{H} \hookrightarrow \mathbb{B}_2$ is Hilbert-Schmidt.

Moreover, if $R : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ is a μ_X -measurable linear operator whose proper linear version R_0 coincides with T on H_{μ_X} , then $R = \widehat{T}$ μ_X -a.e.

Proof The proof follows the well-known lines (e.g. [6], Theorem 3.7.6). For completeness, we provide a sketch of the proof.

First, we verify that any two proper linear measurable mappings $S_1, S_2 : \mathbb{B}_1 \rightarrow \mathbb{B}_1$ that coincide on the Cameron-Martin space of X coincide μ_X -a.s. on \mathbb{B}_1 .

The space \mathbb{B}_2 is separable and reflexive. Therefore, there exists a countable subset G of \mathbb{B}_2^* that separates the points of \mathbb{B}_2 . We only need to verify that

$$\langle S_1 x, b^* \rangle = \langle S_2 x, b^* \rangle$$

μ_X -almost surely for every $b^* \in G$.

The both mappings $x \mapsto \langle S_i x, b^* \rangle$, $i = 1, 2$, are linear and measurable functionals, and they coincide on the Cameron-Martin space. Therefore, they coincide μ_X -almost surely (see Theorem 2.10.7 in [6]). Hence the two proper linear measurable mappings coincide μ_X -almost surely.

Next, we verify that the series is μ_X -almost surely convergent. Since the inclusion mapping of \mathbb{H} into \mathbb{B}_2 is Hilbert-Schmidt, also the mapping $T : H_{\mu_X} \rightarrow \mathbb{B}_2$ is Hilbert-Schmidt. By definition,

$$\sum_{k=1}^{\infty} \|T(C_X \widehat{e}_k)\|_{\mathbb{B}_2}^2 < \infty$$

for any orthonormal basis $\{C_X \widehat{e}_k\}$ of H_{μ_X} .

Then the random series (12) is μ_X -a.s. convergent, since the sum of variances $\sum_{k=1}^{\infty} \|T C_X e_k\|_{\mathbb{B}_2}^2$ converges (e.g [17], Theorem 2 in Chapter 3). Indeed, the coefficient $b \mapsto \widehat{e}_k(b)$ of $T C_X e_k$ in the series (12) are normal random variables on the Lebesgue's completion of the probability space $(\mathbb{B}_1, \mathcal{B}(\mathbb{B}_1), \mu_X)$ by Lemma 1. Moreover, they are independent since $\{\widehat{e}_k\}$ is orthonormal.

The set $L \subset \mathbb{B}_1$, where the series (12) converges is μ_X -measurable linear subspace of full measure. Moreover, $\widehat{T} : L \rightarrow \mathbb{B}_2$ is linear since T is linear. We extend \widehat{T} linearly onto \mathbb{B}_1 by taking such a linear subspace M of \mathbb{B}_1 that \mathbb{B}_1 is a direct algebraic sum of L and M and defining $\widehat{T}(b + b') := \widehat{T}b$ for $b \in L$ and $b' \in M$. Since the convergence holds a.s., the mapping $\widehat{T} : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ is measurable with respect to the Lebesgue's completion of the Borel σ -algebra of \mathbb{B}_1 . □

Definition 4 Let T and \widehat{T} be as in Theorem 1. The mapping \widehat{T} is called *measurable linear extension* of T .

Corollary 1 *Let the assumptions in Theorem 1 hold. The following claims hold for measurable linear extension \widehat{T} of $T : H_{\mu_X} \rightarrow \mathbb{H} \hookrightarrow \mathbb{B}_2$.*

- (i) The set $T(H_{\mu_X})$ coincides with the Cameron-Martin space of \widehat{TX} and the mapping $T : H_{\mu_X} \rightarrow H_{\mu_{\widehat{TX}}}$ has unit norm.
- (ii) (Measurable transpose) Let $T^* : \mathbb{B}_2^* \rightarrow (\mathbb{B}_1^*)_{\mu_X}$ denote the transpose of T and let $b^* \in \mathbb{B}_2^*$. Then

$$\langle \widehat{T}b, b^* \rangle_{\mathbb{B}_2, \mathbb{B}_2^*} = T^*b^*(b)$$

for μ_X -a.e. $b \in \mathbb{B}_1$.

- (iii) (Associativity of compositions) Let \mathbb{H}_2 be a separable Hilbert space, whose inclusion into separable Hilbert space \mathbb{B}_3 is Hilbert-Schmidt. When $\widehat{S} : \mathbb{B}_2 \rightarrow \mathbb{B}_3$ is measurable linear extension of the continuous linear mapping $S : H_{\mu_{\widehat{TX}}} \rightarrow \mathbb{H}_2$, then

$$\widehat{STX} = \widehat{S}(\widehat{TX})$$

almost surely.

- (iv) When T is the identity mapping, we have

$$b = \widehat{T}b$$

μ_X -a.e.

Proof (i) The characterization of the elements of the Cameron-Martin space follows as in Theorem 3.7.3 in [6], which also shows that the mapping T has unit norm.

(ii) Note that the mappings $b \mapsto \langle \widehat{T}b, b^* \rangle_{\mathbb{B}_2, \mathbb{B}_2^*}$ and $b \mapsto T^*b^*(b)$ are measurable and have proper linear versions that coincide on H_{μ_X} for $b^* \in \mathbb{B}^*$ by (11). By Theorem 2.10.7 in [6], these measurable functionals coincide μ_X -a.s.

(iii) Both mappings are measurable linear operators. Indeed, \widehat{STX} is well-defined P -measurable mapping, since \widehat{S} is $\mu_{\widehat{TX}}$ -measurable and the set $\{\widehat{TX} \in B\}$ has zero measure whenever $\mu_{\widehat{TX}}(B) = 0$. The linearity on full measure linear subspace follows then from the definition of extension. Considering approximating sequences of measurable linear functionals and (ii), we obtain

$$\langle \widehat{ST}b, \widehat{h} \rangle_{\mathbb{B}_3, \mathbb{B}_3^*} = S^*\widehat{h}(\widehat{T}b) = T^*S^*\widehat{h}(b) = \langle \widehat{ST}b, \widehat{h} \rangle_{\mathbb{B}_3, \mathbb{B}_3^*}$$

μ_X -a.e for each $\widehat{h} \in B_3^*$. Taking \widehat{h} from some countable dense subset of \mathbb{B}_3^* proves the claim.

- (iv) See Theorem 3.5.1 in [6].

□

3 Measurable boundary operators

Theorem 1 allows us to define the measurable linear extensions of the boundary operators

$$Bu = u|_{\partial D}$$

and

$$Bu = \partial_n u|_{\partial D} + \lambda u|_{\partial D}$$

for $u \in H^1(D)$. Here we omit writing out the inclusion mappings.

For simplicity, the sample space of the boundary mapping is taken to be a scale space (for Banach scale spaces, see [18]). In particular, let us denote with $H_{sc}(\partial D)$, the closure of $H^{-\frac{1}{2}}(\partial D)$ with respect to the norm

$$\|u\|_{sc} := \left(\sum_{k=1}^{\infty} k^{-2} (u, f_k)_{H^{-\frac{1}{2}}(\partial D)}^2 \right)^{\frac{1}{2}},$$

where (f_k) is a fixed orthonormal basis of $H^{-\frac{1}{2}}(\partial D)$.

The choices of sample spaces have little effect for the stochastic analysis.

Corollary 2 *Let X be an $H^{-r}(D)$ -valued Gaussian zero mean random variable for some $r \geq 0$. Let $B : H_{\mu_X} \rightarrow H^{-\frac{1}{2}}(\partial D)$ be a continuous linear mapping and let (e_k) be an orthonormal basis of H_{μ_X} . Then*

$$\widehat{B}b = \sum_{k=1}^{\infty} \widehat{e}_k(b) B e_k$$

belongs to $H_{sc}(\partial D)$ for μ_X -a.e. b , the mapping $\widehat{B} : H^{-r}(D) \rightarrow H_{sc}(\partial D)$ is a μ_X -measurable linear operator, and $\widehat{B}X$ is $H_{sc}(\partial D)$ -valued Gaussian random variable that has zero mean and covariance operator $C_{\widehat{B}X}$ satisfying $C_{\widehat{B}X}u = B C_X B^ u$ for all $u \in H^{\frac{1}{2}}(\partial D)$.*

Proof The claim follows from Theorem 1 after we verify that the inclusion of $H^{-\frac{1}{2}}(\partial D)$ into $H_{sc}(\partial D)$ is Hilbert-Schmidt. By definition, this follows from

$$\sum_{k=1}^{\infty} \|f_k\|_{sc}^2 = \sum_{k=1}^{\infty} k^{-2} < \infty.$$

□

Corollary 2 allows us to define the measurable linear extensions of boundary operators.

Definition 5 *Let X be $H^{-r}(D)$ -valued Gaussian zero mean random variable for some $r \geq 0$ whose Cameron-Martin space H_{μ_X} can be continuously included in $H^1(D)$, and let (e_k) be an orthonormal basis of H_{μ_X} .*

The measurable trace of X onto ∂D is the $H_{sc}(\partial D)$ -valued Gaussian zero mean random variable

$$\widehat{Tr}X = \sum_{k=1}^{\infty} \widehat{e}_k(X) e_k|_{\partial D}.$$

A proper linear version of the corresponding mapping \widehat{Tr} is called the μ_X -measurable trace.

Assuming additionally that $\Delta u \in L^2(D)$ for all $u \in H_{\mu_X}$, the measurable normal derivative of X at ∂D is the $H_{sc}(\partial D)$ -valued Gaussian zero mean random variable

$$\widehat{\partial}_n X = \sum_{k=1}^{\infty} \widehat{e}_k(X) \partial_n e_k,$$

where $\partial_n e_k$ denotes the usual conormal derivative of e_k . A proper linear version of the corresponding mapping $\widehat{\partial}_n$ is called the μ_X -measurable normal derivative.

Let us now verify that the μ_X -measurable trace and μ_X -measurable normal derivative are extensions of the usual operations.

Lemma 2 Let \widehat{B} be a μ_X -measurable trace or a μ_X -measurable normal derivative.

- a) If X has a.s. values in $H^1(D)$, then $\widehat{B}X = BX$ almost surely.
 b) If u belongs to the Cameron-Martin space of X and $\widehat{B}u = 0$ in H_{sc} , then $Bu = 0$ in $H^{-\frac{1}{2}}(\partial D)$.

Proof a) The random series

$$X = \sum_{k=1}^{\infty} \widehat{e}_k(X) e_k$$

converges in $H^1(D)$, and we have

$$\widehat{Tr}X = \sum_{k=1}^{\infty} \widehat{e}_k(X) e_k|_{\partial D} = X|_{\partial D}$$

by continuity of the trace operator. Similar result holds for the normal derivative.

b) Since we are dealing with proper linear versions, we have that $\widehat{B} = B$ on the Cameron-Martin space, and hence $Bu = 0$ in H_{sc} . Moreover, $H^{-\frac{1}{2}}(\partial D)$ is dense in H_{sc} . \square

Remark 4 Take X to be $H^{-r}(D)$ -valued Gaussian zero mean random variable. Since the dual $H^r(D)$ of the sample space $H^{-r}(D)$ of X is dense in $(H^r(D))_{\mu_X}$ and $C_X : (H^r(D))_{\mu_X} \rightarrow H_{\mu_X}$ is an isometry by Remark 3, the orthonormal basis (e_k) of H_{μ_X} can be always chosen so that \widehat{e}_k is from the dual $H^r(D)$ of the sample space $H^{-r}(D)$ of X . Then

$$\widehat{e}_k(X) = \langle X, \widehat{e}_k \rangle_{H^{-r}(D), H^r(D)},$$

by (7) which is notationally simpler choice of a proper linear basis.

4 Existence and uniqueness of the solution

A rough description of the Cameron-Martin space of a Gaussian random variable X leads to a crude idea of the regularity of X .

Lemma 3 Let X be a zero mean Gaussian $H^{-r}(D)$ -valued random variable, whose Cameron-Martin space $H(\mu_X)$ can be continuously imbedded into $H^1(D)$. Then X has realizations in $H^{1-d/2-\delta}(D)$ a.s. for each $\delta > 0$.

Proof A Lipschitz domain is an extension domain (see Theorem A.4 in [19]). Hence, we may apply Maurin's theorem, which tells that the inclusion of the $H^1(D)$ into $H^{-r}(D)$ is Hilbert-Schmidt [13]. Hence, also the Cameron-Martin space can be imbedded into $H^{-r}(D)$ by a Hilbert-Schmidt mapping. Hence, X belongs a.s. in H^{-r} \square

We are now ready to formulate the measurable form of the BVP.

Theorem 2 Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, \dot{W} be the Gaussian white noise on D , $r > d/2 - 1$, and $\lambda, \beta > 0$.

There exists a pathwise unique Gaussian zero mean $H^{-r}(D)$ -valued random field X that satisfies the Dirichlet (or Neumann or Robin) BVP in the following sense:

1. the Cameron-Martin space $H(\mu_X)$ of X can be continuously imbedded into $H^1(D)$ and all $h \in H(\mu_X)$ satisfy $\Delta h \in L^2(D)$,

2. the field X satisfies

$$-\Delta X + \lambda X = \dot{W} \quad (13)$$

in the sense of generalized functions, and

3. the field X satisfies the μ_X -measurable boundary condition

$$\widehat{Tr}X = 0 \text{ in } H_{sc}(\partial D) \quad (14)$$

in the Dirichlet case (or

$$\widehat{\partial}_n X = 0 \text{ in } H_{sc}(\partial D) \quad (15)$$

in the Neumann case, or

$$\widehat{\partial}_n X + \beta \widehat{Tr}X = 0 \text{ in } H_{sc}(\partial D) \quad (16)$$

in the Robin case, correspondingly).

At a glance, the above reformulation may seem eccentric. However, it involves typical elements of elliptic BVPs. Namely, the regularity of the desired solution X is explicitly specified. This is done by requiring that (a) the sample space of X is at least in the Sobolev space $H^{-r}(D)$, (b) X has a Gaussian distribution, and (c) the inclusion of the Cameron-Martin space of X into $H^1(D)$ is continuous. The space $H^{-r}(D)$ may seem unnecessary irregular, but this is not a hindrance, since local and global regularity of the solution can be further studied and refined. On the other hand, such a weak condition is easy to verify.

The Gaussianity of the solution is explicitly required in order to apply Cameron-Martin space techniques. In particular, $\widehat{B} : H^{-r}(D) \rightarrow (H^{-\frac{1}{2}}(\partial D))_{sc}$ is the measurable linear extension of the continuous linear operator $B \circ I : H \rightarrow H^{-\frac{1}{2}}(D)$ (see e.g. [6]). The restriction (i) on the Cameron-Martin space is needed for the definition of the normal derivative. In ordinary elliptic BVPs, the boundary trace of $H^1(D)$ -functions is defined as a continuous linear extension of the trace operator defined originally on continuous functions. In the same spirit, the boundary operator is extended from $H^1(D)$ onto aspired solutions. However, the extended boundary operator \widehat{B} appearing in (14), (15), and (16) is no longer required to be continuous but only measurable, which makes $\widehat{B}X$ well-defined generalized random field on the boundary (see Definition 5).

Another significant difference is that \widehat{B} depends on the solution X through its Cameron-Martin space. However, it can be shown that the extension \widehat{B} coincide with the ordinary continuous boundary operator B for L^2 -loads, which are dense in negatively indexed Sobolev spaces and X contributes to assigning probabilities to sets.

Theorem 3 *Let X be as in Theorem 2. Then*

$$X = \widehat{T}\dot{W},$$

where $T : L^2(D) \rightarrow H^1(D)$ is defined by setting $Tf := u$, where

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } D \\ Bu = 0 & \text{on } \partial D, \end{cases} \quad (17)$$

and the boundary operator $Bu = u|_{\partial D}$ in the Dirichlet case or $Bu = \partial_n u|_{\partial D}$ in the Neumann case or $Bu = \partial_n u|_{\partial D} + \beta u|_{\partial D}$ in the Robin case.

Proof (Theorem 2 and 3) We consider only Robin boundary values. Other boundary conditions are handled similarly (for the Neumann case, choose e.g. $\beta = 0$). For the existence of the Gaussian field, we represent white noise \dot{W} as

$$\dot{W} = \sum_{k=1}^{\infty} \widehat{f}_k(\dot{W}) f_k,$$

where (f_k) is an orthonormal basis of $L^2(D)$ (see Corollary 1 (iv)).

Now define T as in Theorem 3. Such T exists according to the well-known theory of elliptic BVPs (e.g. Theorem 4.11 in [19]).

Then the random series

$$X := \sum_{k=1}^{\infty} \widehat{f}_k(\dot{W}) T f_k$$

is convergent in $H^{-s}(D)$ for any $s > d/2$ by Theorem 1 and defines a zero mean Gaussian random field, whose Cameron-Martin space is $T(L^2)$ equipped with the norm

$$\|Tf\|_{H_{\mu_X}} = \|f\|_{L^2(D)}$$

by Corollary 1. Application of the well-known stability estimate

$$\|Tf\|_{H^1(D)} \leq C \|f\|_{L^2(D)}$$

shows that the Cameron-Martin space H_{μ_X} can be continuously included in $H^1(D)$. This shows also that the realizations of X belong to $H^{-r}(D)$ by Lemma 3. Furthermore, the operator $-\Delta + \lambda$ is continuous on distributions, and we have

$$(-\Delta + \lambda)X = \sum_{k=1}^{\infty} \widehat{f}_k(\dot{W}) (-\Delta + \lambda) T f_k = \dot{W}$$

almost surely, since $(-\Delta + \lambda) T f_k = f_k$ for all k . Moreover, by Corollary 1

$$\begin{aligned} \widehat{\partial}_n X + \beta \widehat{Tr} X &= \widehat{\partial}_n \widehat{T} \dot{W} + \beta \widehat{Tr} \widehat{T} \dot{W} = \widehat{\partial}_n \widehat{T} \dot{W} + \beta \widehat{Tr} \widehat{T} \dot{W} \\ &= \sum_{k=1}^{\infty} \widehat{f}_k(\dot{W}) (\partial_n + \beta Tr) T f_k = 0 \end{aligned}$$

almost surely.

To prove the uniqueness, assume that there are two solutions X and \tilde{X} with measurable boundary operators \widehat{B} and $\widetilde{\widehat{B}}$, respectively. We show that the Cameron-Martin space of $X - \tilde{X}$ is then the trivial space $\{0\}$. Since the Cameron-Martin spaces of X and \tilde{X} are included continuously in $H^1(D)$, also the Cameron-Martin space of $X - \tilde{X}$ is included continuously in $H^1(D)$. (Indeed, by Theorem 3.3.4 in [6] it suffices to consider continuous linear forms and apply Cauchy-Schwartz inequality).

By assumption, $(-\Delta + \lambda)(X - \tilde{X}) = 0$ almost surely. Hence, the same holds for all functions $u \in H^1(D)$ that belong to the Cameron-Martin space of $X - \tilde{X}$ by linearity of $-\Delta + \lambda$. Indeed, we may consider countably many continuous linear functionals that separate the points of $H^{-r-2}(D)$ and apply Theorem 2.10.7 in [6]. Moreover, $\widehat{B}u = Bu = \widetilde{\widehat{B}}u$ in $H^{-\frac{1}{2}}(\partial D)$ by Lemma 2. By uniqueness of the deterministic problem $u \equiv 0$. But Cameron-Martin space reduces to $\{0\}$ only if $X = \tilde{X}$ a.s. \square

5 Improvements in global regularity

We first show when the solutions of the BVP's with the white noise load are square integrable. According to the following theorem, the dimension of the space and regularity of the domain have the key role.

Theorem 4 *Let X and T be as in Theorem 3. Then $X \in L^2(D)$ almost surely if and only if $T : L^2(D) \rightarrow L^2(D)$ is a Hilbert-Schmidt operator.*

Proof Consider an orthonormal basis $\{\phi_k\}$ of $L^2(D)$ consisting of smooth functions. Then $\langle X, \phi_k \rangle_{H^{-r}, H^r}$ are well-defined random variables and we can study finiteness of the expectation

$$\mathbb{E} \sum_{k=1}^{\infty} \langle X, \phi_k \rangle^2 = \sum_{k=1}^{\infty} \mathbb{E} \langle \dot{W}, T^* \phi_k \rangle^2 = \sum_{k=1}^{\infty} \|T^* \phi_k\|_{L^2(D)}^2.$$

This shows the claim, since T is a Hilbert-Schmidt operator if and only if T^* is a Hilbert-Schmidt operator. \square

It is often more natural to show that the realizations are actually continuous. Let us equip the Hölder space $C^{0,\alpha}(D)$, $0 < \alpha < 1$ with the usual norm

$$\|f\|_{C^{0,\alpha}} = \sup_{x \in D} |f(x)| + \sup_{x, y \in D, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

Theorem 5 *Let X and T be as in Theorem 3. If the regular solution operator $T : L^2(D) \rightarrow C^{0,\alpha}(\bar{D})$ continuously, then X has a.s. continuous realizations.*

Proof Let (f_k) be an orthonormal basis of $L^2(D)$. As a linear combination of continuous functions, the random fields

$$\hat{T}_N \dot{W} := \sum_{k=1}^N \hat{f}_k(\dot{W}) T f_k$$

have a.s. continuous realizations for all $N \in \mathbb{N}$. Moreover, the a.s. limit $\lim_{N \rightarrow \infty} \hat{T}_N \dot{W}(x)$ exists for each x . Indeed, by the assumptions, the composition of T with pointwise evaluation $\langle \cdot, \delta_x \rangle$ is a continuous linear functional on the Cameron-Martin space of W . Hence, it has a measurable linear extension

$$b \mapsto \sum_{k=1}^{\infty} \hat{f}_k(b) T f_k(x)$$

on the sample space $H^{-r}(D)$ of the white noise. Furthermore, the distributions of the sequences $\hat{T}_N \dot{W}(x)$ are tight on $H^{-r}(D)$ for fixed x . Moreover,

$$\mathbb{E} |\hat{T}_N \dot{W}(x) - \hat{T}_N \dot{W}(y)|^2 = \sup_{\|f\|_{L^2(D)} \leq 1} |T_N f(x) - T_N f(y)|^2 \leq \|T\|_{L^2, C^{0,\alpha}}^2 |x - y|^\alpha,$$

since $T_N f = T \sum_{k=1}^N \langle f, e_k \rangle e_k$ and

$$\|T_N f\|_{C^{0,\alpha}} \leq \|T\|_{L^2, C^{0,\alpha}} \left\| \sum_{k=1}^N \langle f, e_k \rangle e_k \right\|_{L^2} \leq \|T\|_{L^2, C^{0,\alpha}} \|f\|_{L^2}.$$

By Kolmogorov tightness criterium (e.g. see [24] for a nice Besov space proof), the limit $\hat{T} \dot{W} = \sum_{k=1}^{\infty} \hat{f}_k(\dot{W}) T f_k$ has a.s. continuous realizations. \square

The following familiar examples demonstrate the application of the above theorem.

Example 1 1. When $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain, then there exists such $0 < \alpha < \frac{1}{2}$ that

$$\|Tf\|_{H^{1+\alpha}(D)} \leq C\|f\|_{L^2(D)}$$

in Dirichlet boundary value problems (see [23]). The embedding $H^{1+\alpha} \hookrightarrow C^{0,\alpha}(\overline{D})$ is continuous. Hence, $X = \hat{T}\dot{W}$ has a.s. continuous realizations.

2. When $D \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded convex Lipschitz domain then

$$\|Tf\|_{H^2(D)} \leq C\|f\|_{L^2(D)}$$

for the Dirichlet problem (e.g [16], Theorem 3.2.1.2). The embedding $H^2(D) \hookrightarrow C^{0,\alpha}(\overline{D})$ is continuous for $0 < \alpha < 1/2$. Hence, the assumptions are satisfied and $X = \hat{T}\dot{W}$ has a.s. continuous realizations.

3. When $D \subset \mathbb{R}^d$, $d = 2, 3$, then

$$\|Tf\|_{C^{0,\alpha}} \leq C\|f\|$$

in the case of Robin or Neumann boundary conditions (see [20], Theorem 3.14).

6 Approximations

At this point, it is yet unclear if the generalizations of the Neumann and Robin boundary values have any more value than mathematical eccentricity. However, we show now that when white noise is replaced with its regular approximations, the corresponding approximative solutions converge to the solution of the generalized problem. This clarifies the generalizations from a practical point of view. In the same spirit, we study convergence of Galerkin approximations of high-dimensional problems in Theorem 7. We emphasize that such approximations are interesting, for example, as priors in numerical Bayesian estimation of unknown multivariable functions [21]. Finally, we consider in this section some low-dimensional problems as examples.

The first convergence theorem concerns approximating the white noise only. From various possible approximations of the white noise, we first choose the truncated sums in the measurable linear extension of the identity mapping. However, the proof only requires L^2 -convergence, and transfers therefore to a wider class of approximations.

We use the generic notation B for any boundary operator appearing in Theorem 2.

Theorem 6 Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, \dot{W} be the Gaussian white noise on D , (f_k) be an orthonormal basis of $L^2(D)$, $r > d/2 - 1$ and let X be as in Theorem 2.

If $H^1(D)$ -valued random fields $X^{(m)}$, $m \in \mathbb{N}$, satisfy

$$\begin{cases} -\Delta X^{(m)} + \lambda X^{(m)} = \dot{W}^{(m)} & \text{in } D \\ BX^{(m)} = 0 & \text{on } \partial D, \end{cases}$$

where $\dot{W}^{(m)} = \sum_{k=1}^m f_k(\dot{W})f_k$, then $X^{(m)}$ converges to X in $L^2(\Omega, \Sigma, P; H^{-r}(D))$ as $m \rightarrow \infty$.

Proof The realizations of the load $\dot{W}^{(m)}$ are almost surely in $L^2(D)$. Hence, it is easy to see that the unique field $X^{(m)}$ exists for every $m \in \mathbb{N}$.

Denote with $Tf := u$ the solution of

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } D \\ Bu = 0 & \text{on } \partial D. \end{cases} \quad (18)$$

Then $X = \widehat{T}\dot{W}$ almost surely by Theorem 3. Moreover, $T : L^2(D) \rightarrow H^1(D)$ continuously and the mapping $T : L^2(D) \rightarrow H^{-r}(D)$ is Hilbert-Schmidt (see [13]).

Denote with $P_m : L^2(D) \rightarrow L^2(D)$ the orthogonal projection onto the subspace spanned by $\{f_1, \dots, f_m\}$. Then $X^{(m)} = T\dot{W}^{(m)} = T\widehat{P}_m\dot{W} = \widehat{T}\widehat{P}_m\dot{W}$ by Corollary 1. Denote with (g_ℓ) the orthonormal basis of $H_0^r(D)$, which we identify with the dual space of $H^{-r}(D)$. By Corollary 1,

$$\begin{aligned} \mathbb{E}\|X^{(m)} - X\|_{H^{-r}(D)}^2 &= \mathbb{E} \sum_{\ell=1}^{\infty} \langle X^{(m)} - X, g_\ell \rangle_{H^{-r}(D), H_0^r(D)}^2 \\ &= \sum_{\ell=1}^{\infty} \mathbb{E} (P_m - I) T^* g_\ell (\dot{W})^2 \\ &= \sum_{\ell=1}^{\infty} \|(P_m - I) T^* g_\ell\|_{L^2}^2, \end{aligned}$$

which converges to zero as $m \rightarrow \infty$ by Lebesgue's dominated convergence, since $T : L^2(D) \rightarrow H^{-r}(D)$ is a Hilbert-Schmidt operator and

$$\|(P_m - I) T^* g_\ell\|_{L^2(D)} \leq \|T^* g_\ell\|_{L^2(D)}.$$

□

Next, we study different approximations of the problem arising from Ritz-Galerkin methods. As a preliminary step, we clarify connections between certain measurable linear forms and L^2 -regular approximations of the white noise. We anticipate FEM by indexing the Galerkin subspaces with $h > 0$, which is typically connected to the size of elements in a finite element mesh.

Lemma 4 *Let $Q_h : L^2(D) \rightarrow L^2(D)$ be the orthogonal projection onto a finite-dimensional linear subspace V_h of $L^2(D)$. Then $\widehat{Q}_h\dot{W} \in V_h$ almost surely.*

Proof Let $\{e_k\}$ be an orthonormal basis of $L^2(D)$ whose K first components span V_h . Then

$$\widehat{Q}_h\dot{W} = \sum_{k=1}^K \widehat{e}_k(\dot{W}) e_k \in V_h$$

almost surely.

□

The following theorem demonstrates how convergence of Ritz-Galerkin approximations reduces to convergence of regular cases. Here the Ritz-Galerkin approximations always include also approximations of the white noise with orthogonal projections as in Lemma 4. We focus on Neumann and Robin problems, which have not been studied before.

Theorem 7 Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, \dot{W} be the Gaussian white noise on D , $\lambda, \beta > 0$, $r > d/2 - 1$, and let X be as in Theorem 2 in the case of Robin boundary value.

Let V_h be a finite-dimensional linear subspace of $H^1(D)$, and let X_h be a V_h -valued random variable that satisfies

$$\int_D \nabla X_h \cdot \nabla \phi dx + \lambda \int_D X_h \phi dx + \beta \int_{\partial D} X_h \phi d\sigma = \int_D (\hat{Q}_h \dot{W}) \phi dx \quad (19)$$

for all $\phi \in V_h$, where Q_h is the orthogonal projection onto V_h in $L^2(D)$. Then

$$\mathbb{E} \|X - X_h\|_{H^{-r}(D)}^2 \leq C \sup_{\|f\|_{L^2(D)} \leq 1} \|u^f - u_h^f\|_{H^1(D)}^2, \quad (20)$$

where u^f is the solution of

$$\begin{cases} -\Delta u^f + \lambda u^f = f & \text{in } D \\ \partial_n u^f|_{\partial D} + \beta u^f|_{\partial D} = 0 & \text{on } \partial D, \end{cases} \quad (21)$$

and $u_h^f \in V_h$ is the Ritz-Galerkin approximation defined by

$$\int_D \nabla u_h^f \cdot \nabla \phi dx + \lambda \int_D u_h^f \phi dx + \beta \int_{\partial D} u_h^f \phi d\sigma = \int_D \phi f dx \quad (22)$$

for all $\phi \in V_h$.

Proof By Lemma 4, the measurable linear forms $\hat{Q}_h \dot{W} \in V_h$ a.s. By regularity of the load $\hat{Q}_h \dot{W}$, the standard proof of existence and uniqueness of Ritz-Galerkin approximations holds also for X_h . Moreover, we can write $X_h = T_h(\hat{Q}_h \dot{W})$, where T_h takes regular loads from $L^2(D)$ to corresponding Ritz-Galerkin approximations in V_h .

By Corollary 1, $T_h \hat{Q}_h \dot{W} = \widehat{T_h \hat{Q}_h \dot{W}}$, which almost surely coincides with $\hat{T}_h \dot{W}$ since $\langle \hat{Q}_h f, \phi \rangle = \langle f, \phi \rangle$ for all $f \in L^2(D)$ and $\phi \in V_h$ and, by uniqueness of the solution, $T_h \hat{Q}_h = \hat{T}_h$ on $L^2(D)$.

Denote with $\{\phi_k\}$ an orthonormal basis in $H^r(D)$, and with $\|L\|_{HS:V_1 \rightarrow V_2}$ the Hilbert-Schmidt norm of a linear mapping L from a separable Hilbert space V_1 into a separable Hilbert space V_2 . Then by Corollary 1 (ii),

$$\begin{aligned} \mathbb{E} \|X - X_h\|_{H^{-r}(D)}^2 &= \sum_{k=1}^{\infty} \mathbb{E} \langle X - X_h, \phi_k \rangle_{H^{-r}(D), H^r(D)}^2 \\ &= \sum_{k=1}^{\infty} \|(T^* - T_h^*) \phi_k\|_{L^2(D)}^2 \\ &= \|T^* - T_h^*\|_{HS:H^r(D) \rightarrow L^2(D)}^2 \\ &\leq \|T^* - T_h^*\|_{H^{-1}(D), L^2(D)} \|I\|_{HS:H^r(D) \rightarrow H^{-1}(D)}, \end{aligned} \quad (23)$$

where the embedding I of $H^r(D)$ into $H^{-1}(D)$ is Hilbert-Schmidt by Maurin's theorem. Moreover,

$$\|T^* - T_h^*\|_{H^{-1}(D), L^2(D)} = \|T - T_h\|_{L^2(D), H^1(D)} = \sup_{\|f\|_{L^2(D)} \leq 1} \|u^f - u_h^f\|_{H^1(D)}.$$

□

Corollary 3 *Let the assumptions of Theorem 7 hold. If the finite-dimensional subspaces V_h fulfill the condition*

$$\lim_{h \rightarrow 0} \min_{v \in V_h} \|u - v\|_{H^1(D)} = 0$$

for all $u \in H^1(D)$, then X_h converges to X in $L^2(\Omega, \Sigma, P; H^{-r}(D))$ as $h \rightarrow 0$.

Proof Let u^f and u_h^f satisfy (21) and (22), respectively. By Cea's lemma

$$\begin{aligned} \sup_{\|f\|_{L^2(D)} \leq 1} \|u^f - u_h^f\|_{H^1(D)} &\leq \sup_{\|f\|_{L^2(D)} \leq 1} \min_{v \in V_h} \|u^f - v\|_{H^1(D)}^2 \\ &= \sup_{\|f\|_{L^2(D)} \leq 1} \|(I - Q_h)u^f\|_{H^1(D)}, \end{aligned}$$

where Q_h is the orthogonal projection in $H^1(D)$ onto V_h .

By H^1 -regularity of the solution u^f of the Robin BVP, the corresponding Neumann problem has boundary data $-\beta u^f|_{\partial D} \in H^{\frac{1}{2}}(\partial D)$. By Theorem 4 in [23], the unique solution u^f of this Neumann problem belongs to $H^{1+s}(D)$ for some $0 < s < \frac{1}{2}$.

The embedding of $H^{1+s}(D) \rightarrow H^1(D)$ is compact. Under the mapping $A := f \mapsto u^f$, the image of the unit ball of $L^2(D)$ is relatively compact in $H^1(D)$. For fixed $\varepsilon > 0$, all open balls $B(\phi, \varepsilon)$ of $H^1(D)$ whose center points ϕ belong to the image of the unit ball under the mapping A , form a cover of the image and we may choose a finite subcover consisting of sets $B(\phi_k, \varepsilon)$, $k = 1, \dots, K$. For any f satisfying $\|f\|_{L^2(D)} \leq 1$, we have

$$\begin{aligned} \|(I - Q_h)u^f\|_{H^1(D)} &\leq \|u^f - \phi_k\|_{H^1(D)} + \sum_{k=1}^K \|(I - Q_h)\phi_k\|_{H^1(D)} \\ &\leq \varepsilon + \sum_{k=1}^K \|(I - Q_h)\phi_k\|_{H^1(D)} \end{aligned}$$

for some $k = 1, \dots, K$. □

Remark 5 Replacing $H^1(D)$ with $H_0^1(D)$ in Theorem 7 gives the same results for the homogeneous Dirichlet problem (see [23] for the required regularity).

Many practical applications of boundary value problems involve finite element methods. Below, we demonstrate how to estimate the speed of convergence of finite element approximations. For simplicity, we consider only two dimensional polygonal domains with triangular elements. For details of finite element methods we refer to [8].

The following corollary shows how the speed of convergence in finite element approximations depends on the regularity of the solution. We emphasize that the domain is allowed to be nonconvex.

Corollary 4 *Let X be as in Theorem 2 for $d = 2$. Let \mathcal{T}_h be a triangulation of a bounded polygonal domain $D \subset \mathbb{R}^2$ into elements of diameter at most $h > 0$ and let $V_h \subset H^1(D)$ be the corresponding finite element space consisting of continuous functions. Let the interpolant I_h satisfy $\|u - I_h u\|_{H^1(D)} \leq Ch \|u\|_{H^2(D)}$. Then there exists $0 < s < \frac{1}{2}$ such that the finite element approximations X_h of the solution X satisfy*

$$\mathbb{E} \|X - X_h\|_{H^{-r}(D)}^2 \leq Ch^{2s}.$$

Proof The solution u^f to the homogeneous elliptic boundary value problem (18) belongs to $H^{1+s}(D)$ for some $s > 0$ [23] and satisfies

$$\|u^f\|_{H^{1+s}(D)} \leq c\|f\|_{L^2(D)},$$

for all $f \in L^2(D)$. The assumptions imply then that (e.g. [8], Chapter 12.3)

$$\|u^f - u_h^f\|_{H^1(D)} \leq Ch^s \|u^f\|_{H^{1+s}(D)},$$

where

$$\|u^f\|_{H^{1+s}(D)} \leq C\|f\|_{L^2(D)}.$$

The last claim follows then from (20). \square

Next, we improve the previous convergence results by replacing $H^{-r}(D)$ spaces with the space of continuous functions $C(\overline{D})$ equipped with the usual supremum norm. Instead of L^2 -convergence, we study weak convergence of probability distributions. We apply the notations appearing in Theorem 7.

Theorem 8 *Let $D \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Let X be as in Theorem 2 and let X_h be its finite element approximations. Assume that the solution u^f of (18) and its finite element approximations u_h^f are such that*

$$\lim_{h \rightarrow 0} \sup_{\|f\|_{L^2(D)} \leq 1} |u_h^f(x) - u^f(x)| = 0$$

for any $x \in \overline{D}$ and there exists $\alpha, C > 0$ such that

$$\sup_{\|f\|_{L^2(D)} \leq 1} |u_h^f(x) - u_h^f(y)| \leq C|x - y|^\alpha \quad (24)$$

for all $0 < h < 1$ and $x, y \in \overline{D}$.

Then the probability distribution of X_h converges to the probability distribution of X weakly in the sense of measures on $C(\overline{D})$.

Proof It is clear from (24) that X_h has continuous realizations.

We show that the finite-dimensional probability distributions of X_h converge to the finite-dimensional probability distributions of X , and that the family of probability distributions of X_h on $C(\overline{D})$ is tight.

The convergence of finite-dimensional distributions of zero mean Gaussian random variables $(X_h(x_1), \dots, X_h(x_k))$ depends only on the convergence of the covariances

$$\mathbb{E}X_h(x)X_h(y).$$

Note that $X_h(x) = \langle X_h, \delta_x \rangle$. Moreover,

$$X_h = \widehat{T}_h \dot{W}, \quad (25)$$

where $T_h f$ solves the variational boundary value problem for $f \in L^2(D)$. Hence

$$\mathbb{E}X_h(x)X_h(y) = \mathbb{E}\langle X_h, \delta_x \rangle \langle X_h, \delta_y \rangle = \mathbb{E}\langle \dot{W}, T_h^* \delta_x \rangle \langle \dot{W}, T_h^* \delta_y \rangle = \langle T_h^* \delta_x, T_h^* \delta_y \rangle.$$

We show that $T_h^* \delta_x$ converge to $T^* \delta_x$ in $L^2(D)$. Indeed,

$$\|T_h^* \delta_x - T^* \delta_x\|_{L^2(D)} = \sup_{\|f\|_{L^2(D)} \leq 1} |u_h^f(x) - u^f(x)|$$

which converges to zero by the assumptions. Therefore, the finite dimensional distributions converge.

Next, we verify tightness by Kolmogorov's theorem (see [24]). By linearity

$$X_h(x) - X_h(y) = \langle X_h, \delta_x - \delta_y \rangle. \quad (26)$$

We insert (25) and (26) into

$$\begin{aligned} \mathbb{E}|X_h(x) - X_h(y)|^2 &= \mathbb{E}\langle X_h, \delta_x - \delta_y \rangle^2 = \mathbb{E}\langle \tilde{W}, T_h^*(\delta_x - \delta_y) \rangle^2 \\ &= \|T_h^*(\delta_x - \delta_y)\|_{L^2(D)}^2. \end{aligned}$$

By the assumptions,

$$\mathbb{E}|X_h(x) - X_h(y)|^2 = \sup_{\|f\|_{L^2(D)} \leq 1} |u_h^f(x) - u_h^f(y)|^2 \leq C|x - y|^{2\alpha}.$$

Similarly $\mathbb{E}|X_h(x)|^2 \leq C$. The identification of the limit with the probability distribution of X is carried out with characteristic functions. \square

7 Conclusions

The presented methodology is an effective tool tailored for Gaussian problems and does not directly generalize to nonlinear elliptic problems. However, the principle idea of replacing the normal derivative at the boundary with a measurable mapping may carry over to more general problems.

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